PATH INTEGRALS FOR A CLASS OF P-ADIC SCHRÖDINGER EQUATIONS

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Abstract. The theme of doing quantum mechanics on all abelian groups goes back to Schwinger and Weyl. If the group is a vector space of finite dimension over a non-archimedean locally compact division ring, it is of interest to examine the structure of dynamical systems defined by Hamiltonians analogous to those encountered over the field of real numbers. In this letter a path integral formula for the imaginary time propagators of these Hamiltonians is derived.

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1. Introduction

General formulations of quantum theory when the configuration space is an abelian group have been the theme of studies by Weyl[1] and Schwinger[2]. The present letter arose out of studies of quantum systems not only over the reals but over other fields and rings, especially p-adic fields and adele rings[3,4,5]. The main result is that for an interesting class of Hamiltonians H over nonarchimedean fields that are analogous to the conventional ones, we can set up a formalism that leads to a path integral formula for the propagators $e^{-tH} (t > 0)$. The integrals are over the so-called *Skorokhod space*[6,7] of paths which allow discontinuities, but only of the first kind, namely that the left and right limits exist at all time points and the paths are right continuous. This is to be contrasted with the real case where the path integrals are with respect to conditional Wiener measures and so are on the space

of continuous paths.

The study of quantum systems over finite and discrete structures has been of interest for a long time[8]. Interest in quantum structures over p-adic fields also goes back a long way[9], but in recent years there has been quite a bit of activity, not only over p-adic fields but over the adele rings of number fields also, and there are many treatments of these more general dynamical systems including path integral formulations[10,11]. The literature is extensive and an excellent review that includes a very good exposition of the basics of p-adic theory is[12]. But the formalism presented in this letter appears to be new.

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2. Vector spaces over local fields and division rings[13]

We consider as configuration spaces vector spaces over a division ring D which is finite dimensional over a local (=locally compact, nondiscrete, commutative) field K of arbitrary characteristic. Unlike the case of the real field over which there are only three division algebras of finite rank, namely, the reals, complexes, and the quaternions, the theory of division algebras over nonarchimedean local fields is much richer and is deeply arithmetical. Let K be any nonarchimedean local field of arbitrary characteristic and D a division algebra of finite dimension over K. We shall assume that K is the center of D; this is no loss of generality since we may always replace K by the center of D. Let dx be a Haar measure on D and $|\cdot|$ the usual modulus function on D: $d(ax) = |a|dx$ $(a \neq 0)$, $|0| = 0$. Then $|\cdot|$ is a multiplicative norm which is ultrametric (i.e., $|x+y| \leq \max(|x|, |y|)$) that induces the original topology; and if we define $R = \{x \in D \,|\, |x| \leq 1\}, P = \{x \in D \,|\, |x| < 1\}$, then R is the maximal compact subring of D, P is its maximal left, right, or two-sided ideal, $F := R/P$ is a finite field of, say q, elements, and there is an element π such that $P = R\pi = \pi R$. Put $P^m = \pi^m R = R\pi^m (m \in \mathbf{Z})$ and write $d_0 x$ for the Haar measure for which $\int_R d_0 x = 1$. For any nontrivial additive character χ of D we write δ for the *conductor* of χ; this is the integer characterized by $\chi\big|_{P^{-\delta}} \equiv 1, \quad \chi\big|_{P^{-\delta-1}} \not\equiv 1$. It follows from this that

$$\int_{|x| \leq q^m} \chi(x) d_0 x = \begin{cases} 0, & \text{if } m \geq \delta + 1 \\ q^m, & \text{if } m \leq \delta \end{cases} \tag{1}$$

Let W be a left vector space of finite dimension over D. By a *D-norm* on W is meant a function $|\cdot|$ from W to the nonnegative reals such that (i) $|v| = 0$ if and

only if $v = 0$ (ii) $|av| = |a||v|$ for $a \in D$ and $v \in W$ (iii) $|\cdot|$ satisfies the ultrametric inequality, i.e., $|u+v| \leq \max(|u|, |v|)$ $(u, v \in W)$. The norm on the dual W^* of W is also a D-norm. If we identify W and W^* with D^n by choosing dual bases, and define, for suitable constants $c_i > 0$, $|x| = \max_{1 \leq i \leq n}(c_i|x_i|)$ $(x = (x_1, x_2, \dots, x_n) \in W)$, it is immediate that $|\cdot|$ is a D-norm on W and $|\xi| = \max_{1 \leq i \leq n}(c_i^{-1}|\xi_i|)$ $(\xi = (\xi_1, \dots, \xi_n) \in W^*)$ defines the dual norm on W^*. It is known[13] that every D-norm is of this form. The set of values of $|\cdot|$ on $W \setminus (0)$ is an ordered set

$$0 \ldots < a_{-r} < a_{-r+1} < \ldots < a_{-1} < a_0 < a_1 < \ldots < a_s < \ldots \tag{2}$$

where for some integer $m \geq 1$, $q^{\frac{1}{m}} \leq \frac{a_{r+1}}{a_r} \leq q$ $(r \in \mathbf{Z})$ so that

$$a_0 q^{\frac{r}{m}} \leq a_r \leq a_0 q^r \qquad (r \geq 0) \qquad a_0 q^r \leq a_r \leq a_0 q^{\frac{r}{m}} \qquad (r \leq 0) \tag{3}$$

It is easy to see that there is a constant $A \geq 1$ such that

$$\frac{1}{A} a^n \leq \text{ meas } (\{v||v| \leq a\} \leq A a^n \qquad \forall a > 0 \tag{4}$$

A *D-lattice* in W is a compact open R-submodule of W; these are the sets of the form $\oplus_{1 \leq i \leq n} R e_i$ $((e_i)_{1 \leq i \leq n}$ is a basis for $W)$. For any $u > 0$, the set $\{v||v| \leq u\}$ is a D-lattice. For $x \in W, \xi \in W^*$, let $x\xi$ be the value of ξ at x. For any D-lattice L in W its *dual* lattice $L^*$is the set of all $\xi \in W^*$ such that $x\xi \in R$ for all $x \in L$. If L is as above and $(\varepsilon_i)_{1 \leq i \leq n}$ is the basis of W^* dual to $(e_i)_{1 \leq i \leq n}$, then $L^* = \oplus_{1 \leq i \leq n} \varepsilon_i R$. If $L = \{x \mid |x| \leq u\}$, then

$$\{x \mid |x| \leq u^{-1}\} \subset L^* \subset \{x \mid |x| < q u^{-1}\} \tag{5}$$

Indeed, the first inclusion is clear from $|x\xi| \leq |x||\xi|$. For the second, let $\xi \in L^*$ and choose $x_0 \in W \setminus (0)$ such that $|x_0\xi| = |x_0||\xi|$. Replacing x_0 by $\pi^r x_0$ for $r >> 0$ we may assume that $x_0 \in L$ and $\pi^{-1}x_0 \notin L$. Then $1 \geq |x_0||\xi| > q^{-1}u|\xi|$ so that $|\xi| < qu^{-1}$.

Fix a nontrivial additive character χ on D. Let $\mathcal{S}(W)$ be the Schwartz-Bruhat space of complex-valued locally constant functions with compact supports on W. Let dx be a Haar measure on W. Then $\mathcal{S}(W)$ is dense in $L^2(W, dx)$, and the Fourier transform $\mathbf{F}$ is an isomorphism of $\mathcal{S}(W)$ with $\mathcal{S}(W^*)$, defined by

$$\mathbf{F}(g)(\xi) = \int \chi(x\xi) g(x) dx \quad (\xi \in W^*)$$

For a unique choice of Haar measure $d\xi$on W^* (the *dual measure*) we have,

$$g(x) = \int \chi(-x\xi) \mathbf{F} g(\xi) d\xi \quad (x \in W, g \in \mathcal{S}(W))$$

If $W = D^n = W^*$ and $dx = q^{-n\delta/2} d_0x_1 \dots d_0x_n$, then $d\xi = q^{-n\delta/2} d_0\xi_1 \dots d_0\xi_n$ is the dual measure.

3. Hamiltonians over W

Consider the p-adic Schrödinger theory which consists of the study of the spectra of and semigroups generated by operators ("Hamiltonians") in $L^2(W)$ of the form $H = H_0 + V$. Here H_0 is a pseudodifferential operator and V is a multiplication operator. Write $M_{W,b}$ for multiplication by $|x|^b (b > 0)$ in $\mathbf{H} = L^2(W)$, and put $\Delta_{W,b} = \mathbf{F} M_{W,b} \mathbf{F}^{-1}$. We consider Hamiltonians will of the form

$$H_{W,b} = \Delta_{W,b} + V$$

Notice that for $D = \mathbf{R}$ and $b = 2$ this construction gives $H_{W,b} = -\Delta$.

4. The probability densities $f_{t,b}$

We shall show that the dynamical semigroup $e^{-tH_{W,b}} (t > 0)$ is just convolution by a one-parameter semigroup of *probability densities*. Lemma 2 contains the key calculation in the present letter and allows us to replace the Gaussian densities of the conventional theory by these densities. 1_E is the characteristic function of the set E.

LEMMA 1. *Fix dual Haar measures dx and $d\xi$ on W and W^* respectively. Let L be a D-lattice in W^*. Then $\mathbf{F}1_L =$ meas $(L)1_{\pi^{-\delta}L^*}$. In particular,*

$$\int_L \chi(x\xi) d\xi \geq 0 \quad (x \in W)$$

Moreover, if $L = \{\xi \mid |\xi| \leq u\}$ where $u > 0$, then

$$\int_{|\xi| \leq u} \chi(x\xi) d\xi = \begin{cases} \textit{meas } (L) & \textit{if } |x| \leq q^\delta u^{-1} \\ 0 & \textit{if } |x| \geq q^{1+\delta} u^{-1} \end{cases}$$

Proof. This is standard. First assume that $W = D = W^*, L = R = L^*$. Then $d\xi =$ meas $(R) d_0\xi$. Now $\int_R \chi(x\xi) d_0\xi = |x|^{-1} \int_{|\zeta| \leq |x|} \chi(\zeta) d_0\zeta$ for $x \neq 0$ so that

$$\int_R \chi(x\xi) d_0\xi = 1_{P^{-\delta}}(x)$$

from (2.1). The result for general W, L is immediate since we may suppose that $W = D^n = W^*, L = R^n = L^*$. The last assertion of the lemma follows from (2.5).

LEMMA 2. *Fix $t > 0$ and $b > 0$ and let W be a n-dimensional left vector space over D with a D-norm $|\cdot|$. Then the function φ on W^* defined by*

$$\varphi(\xi) = \exp(-t|\xi|^b) \quad (\xi \in W^*)$$

is in $L^m(W, d\xi)$ for all $m \geq 1$ and is the Fourier transform of a continuous probability density f on W with $f(ax) = f(x)$ for $x \in W, a \in D, |a| = 1$. Moreover (i) $0 < f(x) \leq f(0) \leq A\ t^{-n/b}$ for all $t > 0, x \in W$, A being a constant > 0 not depending on t, x (ii) For $0 \leq k < b$ we have, for all $t > 0$ and a constant $A > 0$ independent of t,

$$\int |x|^k f(x) dx \leq A\ t^{k/b}$$

Proof. From now on A will denote a generic constant > 0 independent of $t > 0, x, \xi$. By (2.3), (2.4) we have, for $t > 0$,

$$\int_{W^*} e^{-t|\xi|^b} d\xi = \sum_{r \in \mathbf{Z}} e^{-ta_r^b} \int_{|\xi| = a_r} d\xi \leq A \sum_{r \in \mathbf{Z}} a_r^n e^{-ta_r^b} < \infty$$

Further

$$\begin{aligned}
\int_{W^*} e^{-t|\xi|^b} d\xi &= \sum_{r \in \mathbf{Z}} e^{-ta_r^b} \int_{|\xi| = a_r} d\xi = \sum_{r \in \mathbf{Z}} e^{-ta_r^b} \int_{|\xi| \leq a_r} d\xi - \sum_{r \in \mathbf{Z}} e^{-ta_r^b} \int_{|\xi| \leq a_{r-1}} d\xi \\
&= \sum_{r \in \mathbf{Z}} \left(e^{-ta_r^b} - e^{-ta_{r+1}^b} \right) \int_{|\xi| \leq a_r} d\xi \leq A \sum_{r \in \mathbf{Z}} a_r^n \left(e^{-ta_r^b} - e^{-ta_{r+1}^b} \right) \\
&\leq A \sum_{r \in \mathbf{Z}} t \int_{a_r^b}^{a_{r+1}^b} e^{-ty} y^{n/b} dy = At \int_0^\infty e^{-ty} y^{n/b} dy = At^{-n/b}
\end{aligned}$$

So $\varphi \in L^m(W^*, d\xi)$ for $m \geq 1$. Set

$$f(x) = \int \chi(x\xi) e^{-t|\xi|^b} d\xi \quad (x \in W)$$

Clearly $f(ax) = f(x)$ for $|a| = 1$. We prove that $f > 0$ and $\in L^1(W, dx)$. As before,

$$f(x) = \sum_{r \in \mathbf{Z}} e^{-ta_r^b} \int_{|\xi| = a_r} \chi(x\xi) d\xi = \sum_{r \in \mathbf{Z}} \left(e^{-ta_r^b} - e^{-ta_{r+1}^b} \right) \int_{|\xi| \leq a_r} \chi(x\xi) d\xi \qquad (1)$$

By Lemma 1, all the terms are ≥ 0 and are > 0 for $r << 0$. Hence $f(x) > 0$ for all $x \in W$. Moreover Lemma 1 and (1) give

$$\begin{aligned}
\int |x|^k f(x)dx &= \sum_{r\in\mathbf{Z}} \left(e^{-ta_r^b} - e^{-ta_{r+1}^b}\right) \int_{|x|<q^{1+\delta}a_r^{-1}} |x|^k dx \int_{|\xi|\leq a_r} \chi(x\xi)d\xi \\
&\leq A \sum_{r\in\mathbf{Z}} \left(e^{-ta_r^b} - e^{-ta_{r+1}^b}\right) a_r^{-k} \int_{|x|<q^{1+\delta}a_r^{-1}} dx \int_{|\xi|\leq a_r} d\xi \\
&\leq A \sum_{r\in\mathbf{Z}} \left(e^{-ta_r^b} - e^{-ta_{r+1}^b}\right) a_{r+1}^{-k} \leq At \int_0^\infty e^{-ty} y^{-k/b} dy \leq At^{k/b}
\end{aligned}$$

This proves in particular that $f \in L^1(W, dx)$ and completes the proof.

Fix $b > 0$ and write $f_{t,b}, \varphi_{t,b}$ for f and φ. It is now clear that the $(f_{t,b})_{t>0}$ form a continuous convolution semigroup of probability densities which goes to the Dirac delta measure at 0 when $t \to 0$. Hence for any $x \in W$ one can associate a W-valued separable stochastic process with independent increments, $(X(t))_{t\geq 0}$, with $X(0) = x$, such that for any $t > 0, u \geq 0$, $X(t+u) - X(u)$ has the density $f_{t,b}$. As usual E_x denotes the expectation value with respect to this process. Clearly, when $b = 2$ and $D = \mathbf{R}$, this is the Wiener process.

5. The paths of the stochastic processes $(X(t))_{t\geq 0}$ and $(X_{T,y}(t))_{t\geq 0}$

Lemma 4.2 may be rewritten as follows.

LEMMA 1. *We have, for any $t > 0$, $E_0|X(t)|^k < \infty (0 \leq k < b)$; and for a fixed k, there is a constant $A_k > 0$ such that $E_0|X(t)|^k \leq A_k t^{k/b}$ for all $t > 0$.*

Let $D([0, \infty) : M)$ be the space of right continuous functions on $[0, \infty)$ with values in the complete separable metric space M having only discontinuities of the first kind. For any $T > 0$ we write $D([0, T] : M)$ for the analogous space of right continuous functions on $[0, T)$ with values in the complete separable metric space M having only discontinuities of the first kind, and left continuous at T. These are the Skorokhod spaces[6,7] mentioned at the beginning.

LEMMA 2. *The process $X(t)_{t\geq 0}$ with $X(0) = x$ has paths in the space $D([0, \infty) : W)$ and is concentrated in the subspace of paths taking the value x for $t = 0$.*

Proof. It is immediate from the preceding proposition that for $0 < t_1 < t_2 < t_3$,

$$\begin{aligned}
E_x\{|X(t_2) - X(t_1)|^k |X(t_3) - X(t_2)|^k\} &= E_0\{|X(t_2) - X(t_1)|^k |X(t_3) - X(t_2)|^k\} \\
&\leq A(t_3 - t_1)^{2k/b} \qquad (1)
\end{aligned}$$

So, if we take k such that $b/2 < k < b$, we may use the criterion of Čentsov[14] to conclude the required result.

We shall now construct the processes obtained from $(X(t))_{t\geq 0}$ by conditioning them to go through y at time $t = T$. The density $f_{t,b}$ is everywhere positive and continuous and so the finite dimensional conditional densities are defined everywhere and allow us to build the conditioned process. We wish to prove that the corresponding probability measures can be defined on the Skorokhod space $D([0,T]:W)$, and that they form a continuous family depending on the starting point x and the finishing point y. This will follow from the Čentsov criteria in the usual manner if we prove the following lemma.

LEMMA 3. *We have, uniformly for all* $0 < t_1 < t_2 < t_3 < T$ *with* $|t_3 - t_1| \leq T/2$, *and* $z \in V$, *and for* $b/2 < k < b$,

$$E_0\{|X(t_2) - X(t_1)|^k |X(t_3) - X(t_2)|^k \ \big| X(T) = z\} \leq A \frac{1}{f_{T,b}(z)} (t_3 - t_1)^{2k/b}$$

Proof. The conditional expectation in question is (writing f_t for $f_{t,b}$)

$$\int |u_2|^k |u_3|^k \frac{f_{t_1}(u_1) f_{t_2 - t_1}(u_2) f_{t_3 - t_2}(u_3) f_{T - t_3}(z - u_1 - u_2 - u_3)}{f_T(z)} du_1 du_2 du_3$$

Since $|t_3 - t_1| \leq T/2$, either t_1 or $|T - t_3|$ is $\geq T/4$, so that one of the two factors $f_{t_1}(u_1)$, $\quad f_{T-t_3}(z - u_1 - u_2 - u_3)$ is bounded uniformly by a constant; the other factor can then be integrated with respect to u_1 and the conditional expectation is majorized by

$$A \frac{1}{f_T(z)} E_0\{|X(t_2) - X(t_1)|^k |X(t_3) - X(t_2)|^k\}$$

and the result follows from (1) and the Čentsov[14] criterion.

The following theorem is now clear.

THEOREM 4. *There are unique families of probability measures* $\mathbf{P}_x^b, \mathbf{P}_{x,y}^{T,b} (x, y \in W)$ *on* $D([0,\infty):W)$ *and* $D([0,T]:W)$ *respectively, continuous with respect to* x *and* (x,y) *respectively, such that* $\mathbf{P}_x^b$ *is the probability measure of the* X*-process starting from* x *at time* $t = 0$, *and* $\mathbf{P}_{x,y}^{T,b}$ *is the probability measure for the* X*-process that starts from* x *at time* $t = 0$ *and is conditioned to pass through* y *at time* $t = T$.

It is now clear following the usual arguments[15] that one can obtain the formula for the propagators. For simplicity let $V \geq 0$ and let the operator $H_{W,b}$ be essentially self-adjoint on $\mathcal{S}(W)$. This is the case if V is bounded, but see also[10].

Feynman–Kac propagator for $e^{-tH_{W,b}}(t > 0)$. The operator $e^{-tH_{W,b}}(t > 0)$ is an integral operator in $L^2(W)$ with kernel $K_{t,b}$ on $W \times W$ which is represented by the following integral on the space $\mathcal{D}_t = D([0,t] : W)$:

$$K_{t,b}(x : y) = \int_{\mathcal{D}_t} \exp\left(-\int_0^t V(\omega(s))ds\right) dP^t_{x,y}(\omega)\cdot f_{t,b}(x-y) \quad (x, y \in W)$$

One dimensional case with $W = D$. Here $f_{t,b}(x)$ depends only on $|x|$ and so is known if we compute the values $f_{t,b}(\pi^{-m})$. We have, using the self-dual $dx = q^{-\delta/2}d_0x$,

$$f_{t,b}(\pi^{-m}) = q^{-\delta/2} \sum_{r \leq -m+\delta} q^r \left(e^{-tq^{rb}} - e^{-tq^{(r+1)b}}\right)$$

Coulomb problem. Over K one should take a three dimensional vector space. The choice that is closest to what happens in the real case is the one where we take the *unique* 4-dimensional division algebra D over K and take for W the subspace of elements of D of trace 0 in the irreducible representation of D in dimension 2 over the separable algebraic closure K_s of K. Let the characteristic of K be $\neq 2$; then D can be described as a *quaternion algebra* generated by "spin matrices" and the analogy with the real case is really close. In fact[13,16] given any two elements $a, b \in K^\times$ such that b is not a norm of an element of $K(\sqrt{a})$ (such a, b exist), one can exhibit D as the ("quaternion algebra") algebra over K with generators i, j and relations

$$i^2 = a, \quad j^2 = b, \quad ij = -ji (=: k)$$

One writes $(a, b)_K$ for this algebra and notes that $(-1, -1)_{\mathbf{R}}$ is just the usual algebra of quaternions. Write $\sqrt{a}$, $\sqrt{-b}$ for square roots of $a, -b$ which are in K_s. If we define

$$\sigma_1 = \begin{pmatrix} 0, & \sqrt{a} \\ \sqrt{a}, & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0, & -\sqrt{-b} \\ \sqrt{-b}, & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \sqrt{a}\sqrt{-b}, & 0 \\ 0, & -\sqrt{a}\sqrt{-b} \end{pmatrix}$$

then $\sigma_1^2 = aI, \quad \sigma_2^2 = bI, \quad \sigma_1\sigma_2 = -\sigma_2\sigma_1 = \sigma_3$, and so there is a faithful irreducible representation ρ of $(a, b)_K$ in dimension 2 such that $\rho(i) = \sigma_1, \rho(j) = \sigma_2, \rho(k) = \sigma_3$. Thus D is the algebra of matrices

$$x = \begin{pmatrix} x_0 + x_3\sqrt{a}\sqrt{-b} & x_1\sqrt{a} - x_2\sqrt{-b} \\ x_1\sqrt{a} + x_2\sqrt{-b}, & x_0 - x_3\sqrt{a}\sqrt{-b} \end{pmatrix} \qquad (x_j \in K)$$

Then $\det(x) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$, and $\det^{1/2}$ is a K-norm on D. If we take W to be the subspace of $x \in D$ with $Tr(x) = 2x_0$ is 0, one can study the Coulomb problem on W. The Hamiltonian is

$$H = \Delta_{D,b} - eM_{\frac{1}{|x|}} \qquad (e > 0 \text{ a constant })$$

where $M_{\frac{1}{|x|}}$ is multiplication by $\frac{1}{|x|}$. This is invariant under the group U of elements of determinant 1 of D which is *semisimple*. We shall treat these matters on a later occasion.

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